On the Goldbach Property for Group Semidomains

Eddy Li, Advaith Mopuri, and Charles Zhang

MIT PRIMES-USA

(Mentored by Dr. Harold Polo)

MIT PRIMES October Conference
12 October 2024

Table of Contents

Monoids

Definition

A pair $(M, +)$ of a set M and a binary operation $(+)$ is a **monoid** if the following conditions hold:

- M is closed under $(+)$,
- $(+)$ is commutative and associative, and
- M exhibits an identity element $0 \in M$ under $(+)$.

Examples (Monoids)

- $(\mathbb{N}_0, +)$, the nonnegative integers under addition.
- $(\mathbb{N}, \cdot)$, the positive integers under multiplication.
- $(\mathbb{Q}_{\geq 0}, +)$, the nonnegative rational numbers under addition.

Special Monoids

Definition

Let $(M, +)$ be a monoid with identity 0, and let $a, b, c \in M$ and $n \in \mathbb{N}$.

- M is **cancellative** if $a + c = b + c$ implies $a = b$.
- M is **torsion-free** if $na = nb$ (repeated addition) implies that $a = b$.
- M is **linearly ordered** if there exists a total order $(\preceq)$ on M such that having $a \preceq b$ implies $a + c \preceq b + c$.
- N is a **submonoid** of another monoid M if $0 \in N \subseteq M$ and N is closed under $(+)$.

Unless specified otherwise, we tacitly assume that all monoids that we shall deal with are cancellative.

Examples (More Monoids)

- $(\mathbb{N}_0, +)$ is cancellative, torsion-free, and linearly ordered.
- $(\{0, 3, 5, 6\} \cup \mathbb{N}_{\geq 8}, +)$ is a submonoid of $(\mathbb{N}_0, +)$, thus inheriting its properties of being cancellative, torsion-free, and linearly ordered.
- $(\mathbb{Z}/6\mathbb{Z}, +)$ is cancellative but not torsion-free, as $1 + 1 \equiv_6 4 + 4$.

Semirings

Eddy Li, Advaith Mopuri, and Charles Zhang

Definition

A triple $(R, +, \cdot)$ is a **semiring** if the following conditions hold:

- $(R, +)$ is a monoid with its identity denoted by 0,
- $(R \setminus \{0\}, \cdot)$ is a semigroup with an identity denoted by 1 with $1 \neq 0$,
- $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$.

A subset R' of a semiring $(R, +, \cdot)$ is a **subsemiring** if $(R', +, \cdot)$ is a semiring when $(+)$ and $(\cdot)$ are restricted to the domain of R'.

Examples (Semirings)

- $(\mathbb{N}_0, +, \cdot)$ is a semiring. In fact, it is a subsemiring of $(\mathbb{Z}, +, \cdot)$.
- $(\mathbb{Q}_{\geq 0}, +, \cdot)$ is a subsemiring of $(\mathbb{R}_{\geq 0}, +, \cdot)$.

Integral Domains

Eddy Li, Advaith Mopuri, and Charles Zhang

Definition

A triple $(R, +, \cdot)$ is an **integral domain** if the following conditions hold:

- $(R, +)$ is an abelian group,
- $(R \setminus \{0\}, \cdot)$ is a cancellative monoid. In other words, $(R \setminus \{0\}, \cdot)$ has no zero divisors, and
- $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$.

Examples (Integral Domains)

- $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are all integral domains under the standard $(+)$ and $(\cdot)$.

Semidomains

Definition

A **semidomain** is a subsemiring of an integral domain.

One may think of semidomains as integral domains in which additive inverses are no longer required for all elements.

Examples (Semidomains)

- $(\mathbb{N}_0, +, \cdot)$ is a semidomain as it is a semiring embedded under the integral domain $(\mathbb{Z}, +, \cdot)$.
- $(\mathbb{N}_0[x^{\pm 1}], +, \cdot)$ is the semidomain containing all Laurent polynomials of the form $f = \sum_{i=0}^{n} c_i x^{k_i}$ where $c_i \in \mathbb{N}_0$ and $k_i \in \mathbb{Z}$ for all i.
- Similarly, $(\mathbb{Z}[x^{\pm 1}], +, \cdot)$ is the integral domain consisting of all Laurent polynomials with coefficients in $\mathbb{Z}$.

  Observe that $(\mathbb{N}_0[x^{\pm 1}], +, \cdot)$ is a subsemiring of the integral domain $(\mathbb{Z}[x^{\pm 1}], +, \cdot)$, so it is a semidomain.

Special Elements of Semidomains

Definition

Let S be any semidomain with $s, a, b, c \in S$.

- Denote the identities of the additive monoid $(S, +)$ and the multiplicative monoid $(S \setminus \{0\}, \cdot)$ by 0 and 1, respectively.
- c is an **additive unit** if there exists $-c \in S$ such that $c + -c = 0$.
- s is an **additive irreducible** if s is not an additive unit and $s = a + b$ implies that either a or b is an additive unit.
- c is a **multiplicative unit** if there exists $c^{-1} \in S$ such that $cc^{-1} = 1$.
- s is a **multiplicative irreducible** if s is not a multiplicative unit and having $s = ab$ implies that either a or b is a multiplicative unit.

We denote these sets as $\mathscr{U}_+(S)$, $\mathscr{A}_+(S)$, $S^\times$, and $\mathscr{A}(S)$, respectively.

Special Semidomains

We are interested in the properties of the additive monoid $(S, +)$ of a given semidomain S.

Definition

- S is **additively reduced** if 0 is the only additive unit of S.
- S is **additively atomic** if $(S, +)$ is atomic, so that all $s \in S \setminus \mathscr{U}_+(S)$ can be written as the sum of finitely many additive irreducibles.
- S is **additively Furstenberg** if $(S, +)$ is Furstenberg, so that all $s \in S \setminus \mathscr{U}_+(S)$ has some additive irreducible additively dividing it.

Examples (More Semidomains)

- $\mathbb{N}_0$ is a semidomain with $\mathscr{A}_+(S) = \{1\}$, $S^\times = \{1\}$, and $\mathscr{A}(S) = \mathbb{P}$. Furthermore, $\mathbb{N}_0$ is additively reduced, additively atomic, and so additively Furstenberg.
- $\mathbb{N}_0[x^{\pm 1}]$ is a semidomain with $\mathscr{A}_+(S) = S^\times = \{x^k \mid k \in \mathbb{Z}\}$, while $\mathscr{A}(S)$ is very complicated. Further, $\mathbb{N}_0[x^{\pm 1}]$ is additively reduced, additively atomic, and so additively Furstenberg.

Motivation

- The Goldbach conjecture was initially presented in a letter from Christian Goldbach to Leonhard Euler (1742), hypothesizing every even integer greater than 2 can be expressed as the sum of two positive prime numbers.
- While still an open problem, progress has been made in other domains other than the original $\mathbb{N}_0$.
- Rather recently, Liao and Polo (2023) showed an analogue of the Goldbach conjecture over the semidomain $\mathbb{N}_0[x^{\pm 1}]$.
- Kaplan and Polo (2023) have then extended this result to all additively reduced and additively atomic Laurent polynomial semidomains $S[x^{\pm 1}]$ satisfying $\mathscr{A}_+(S) = S^\times$.

In this talk, we will try to extend both of the last two theorems presented by Liao-Polo and Kaplan-Polo to more general structures – namely, group semidomains.

Group Semidomains

Definition

Let S be a semidomain and let G be a torsion-group abelian. We define the **group semidomain** $S[G]$ as containing all polynomial expressions of the form

$$f(x) = \sum_{i=0}^{n} s_i x^{g_i}$$

such that $s_i \in S$, and $g_i \in G$ for $0 \leq i \leq n$ and $g_i < g_{i+1}$ for all $0 \leq i < n$.

We require G to be torsion-free and abelian because of Levi's Theorem:

Theorem (Levi, 1913)

For an abelian group G, the following conditions are equivalent.

- G is torsion-free.
- G can be turned into a linearly ordered monoid.

Example (Group Semidomains)

- Let S be a semidomain. Then $S[x^{\pm 1}] = S[\mathbb{Z}]$.

Support

Definition

Let $S[G]$ be a group semidomain. For any polynomial expression $f = \sum_{i=0}^{n} s_i x^{g_i} \in S[G]$, define the **support** of f, which we denote by $\text{supp}(f)$, as

$$\text{supp}(f) := \{g_i \mid s_i \neq 0, 0 \leq i \leq n\}.$$

In this way, note that for $f \in S[G]$, the element f has $|\text{supp}(f)|$ terms.

Examples (Support)

In $\mathbb{N}_0[x^{\pm 1}]$, consider the polynomials

$$f = 1 + x + 2x^3 + x^4 \text{ and}$$
$$g = 2 + 4x + x^3 + 2x^4.$$

Observe that $\text{supp}(f) = \text{supp}(g) = \{0, 1, 3, 4\}$.

Additionally, note that f is multiplicatively irreducible while g is not, as $g = (1 + 2x)(2 + x^3)$ and neither $1 + 2x$ nor $2 + x^3$ are multiplicative units.

Our First Main Result

Theorem (L–M–Z, 202?)

Let S be an additively reduced and additively Furstenberg semidomain, and let G be an abelian torsion-free group. The following statements are equivalent.

1. $\mathscr{A}_+(S) = S^\times$.
2. Every $f \in S[G]$ with $|\text{supp}(f)| > 1$ can be expressed as the sum of at most two multiplicative irreducibles.
3. There exists $k \in \mathbb{N}$ such that every $f \in S[G]$ with $|\text{supp}(f)| > 1$ can be expressed as the sum of at most k multiplicative irreducibles.

Moreover, if any of the previous statements hold and $f \in S[G]$ is not of one of the following forms:

(a) $f = s_0 x^{g_0} + s_1 x^{g_1}$, where either $s_0 \in S^\times$ or $s_1 \in S^\times$, or

(b) $f = s_0 x^{g_0} + s_1 x^{g_1} + s_2 x^{g_2}$, where either $s_0, s_1, s_2 \in S^\times$,

then f can be decomposed into exactly two multiplicative irreducible summands in $S[G]$.

Furstenbergness and Atomicity

Recall the following definitions.

- S is **additively atomic** if $(S, +)$ is atomic, so that all $s \in S \setminus \mathscr{U}_+(S)$ can be written as the sum of finitely many additive irreducibles.
- S is **additively Furstenberg** if $(S, +)$ is Furstenberg, so that all $s \in S \setminus \mathscr{U}_+(S)$ has some additive irreducible additively dividing it.

Additive atomicity automatically implies additive Furstenbergness, but are they truly different criteria?

- A construction of Lin–Rabinovitz–Zhang (2023) yields a monoid that is Furstenberg but not atomic.
- Constructions of Gotti–Polo (2023) and Fox–Goel–Liao (2023) yield semidomains that are multiplicatively Furstenberg but not multiplicatively atomic.

What about additive Furstenbergness and additive atomicity?

Concluding Remarks on Furstenbergness

We construct an infinite class of semidomains which are additively Furstenberg but not additively atomic.

Proposition (L–M–Z, 202?)

For primes $p, q \in \mathbb{P}_{\geq 3}$ satisfying $\frac{q}{p} > \frac{1+\sqrt{5}}{2}$, set

$$(\kappa_{p,q}, \lambda_{p,q}) := \left(\frac{q}{p+q}, \frac{q^2}{p^2+pq} \right).$$

Then the semidomain $S_{p,q} := \mathbb{N}_0\left[\kappa_{p,q}, \lambda_{p,q}\right]$ is additively Furstenberg but not additively atomic. In particular, $\mathscr{A}_+(S_{p,q}) = \{\kappa_{p,q}^n : n \in \mathbb{N}_0\}$.

Highlights of the Proposition

$S_{3,5} = \mathbb{N}_0[\frac{5}{8}, \frac{25}{24}]$ and $S_{7,13} = \mathbb{N}_0[\frac{13}{20}, \frac{169}{140}]$ are additively Furstenberg but not additively atomic.

Observe that p and q can become arbitrarily large. For example, putting $S_{457,977} = \mathbb{N}_0[\frac{977}{1434}, \frac{954529}{655338}]$ works too.

Acknowledgments

- We are grateful to our mentor, Dr. Harold Polo, for introducing us to the field of semidomain theory as well as his invaluable guidance, assistance, and support throughout the research and writing phases.
- We additionally thank Dr. Felix Gotti for his indispensable help and feedback on this presentation.
- We also extend our appreciation to the MIT PRIMES-USA program for creating this research group (isomorphic to $\mathbb{Z}/3\mathbb{Z}$) and providing us with resources and opportunities that would otherwise have been unimaginable.

References

- H. Fox, A. Goel, and S. Liao: *Arithmetic of Semisubtractive Semidomains*, arXiv:https://arxiv.org/pdf/2311.07060.
- F. Gotti and H. Polo: *On the arithmetic of polynomial semidomains*, Forum Math. (2023) **35** 1179–1197.
- D. R. Hayes: *A Goldbach theorem for polynomials with integral coefficients*, Amer. Math. Monthly **72** (1965) 45–46.
- N. Kaplan and H. Polo: *A Goldbach theorem for Laurent series semidomains*, arXiv:https://arxiv.org/abs/2312.14888.
- S. Liao and H. Polo: *A Goldbach theorem for Laurent polynomials with positive integer coefficients*, Amer. Math. Monthly (to appear).
- A. Lin, H. Rabinovitz, and Q. Zhang: *The Furstenberg property in Puiseux monoids*, arXiv:https://arxiv.org/abs/2309.12372.

End of Presentation

THANK YOU!